The Uniform Boundedness Principle

For me, the uniform boundedness principle has always been something surprising and counter-intuitive. I think this is something that happens many times in mathematics, but usually, when you see the proof of the result, you realize where is the key point that solves the paradox. This makes you feel better and then the result becomes expected instead of surprising or counter-intuitive. This is something that did not happen to me when I first saw a proof of the uniform boundedness principle, so I wrote this proof which, in my opinion, gives some intuition of why the uniform boundedness principle is true.

Theorem 1. *Let X be a Banach space and Y a normed vector space. Let $\mathcal{T} \subset \mathcal{L}(X, Y)$ be a family of bounded linear operators such that, for every $x \in X$, there exists $C_x > 0$ such that*

$$\sup_{T \in \mathcal{T}} \|Tx\| \leq C_x.$$

Then, there exists $C > 0$ such that

$$\sup_{T \in \mathcal{T}} \|T\| \leq C$$

Proof. Let us prove it by contradiction. Assume that $\sup_{T \in \mathcal{T}} \|T\| = \infty$. We will find $x \in X$ and $T_n \subset \mathcal{T}$ such that $\|T_n x\| \geq n$ for every $n \in \mathbb{N}$. What we will do is to construct x as a convergent series. We will construct the serie so that the main term of $\|T_n x\|$ will be $\|T_n x_n\|$, and we can neglect the rest of them.

Let us construct the sequence $\{(T_n, x_n)\}_n$ by induction. We first take (x_1, T_1) such that $\|x_1\| \leq 1$, $\|T_1\| \geq 5 \cdot 6$ and $\|T_1 x_1\| = \frac{2}{5}\|T_1\|$. Now, by induction we choose $\|T_n\| \geq 5 \cdot 6^n \cdot n$ and we choose x_n such that

(a) If $\left\|T_n\left(\sum_{i=1}^{n-1} \frac{x_i}{6^i}\right)\right\| \geq \frac{2}{5}\frac{\|T_n\|}{6^n}$ then we choose $x_n = 0$, so we have $\left\|T_n\left(\sum_{i=1}^{n} \frac{x_i}{6^i}\right)\right\| \geq \frac{2}{5}\frac{\|T_n\|}{6^n}$

(b) If $\left\|T_n\left(\sum_{i=1}^{n-1} \frac{x_i}{6^i}\right)\right\| < \frac{2}{5}\frac{\|T_n\|}{6^n}$ we choose $\|x_n\| \leq 1$ such that $\|T_n x_n\| \geq \frac{4}{5}\|T_n\|$. Then,

$$\left\|T_n\left(\sum_{i=1}^{n} \frac{x_i}{6^i}\right)\right\| \geq \left\|T_n \frac{x_n}{6^n}\right\| - \left\|T_n\left(\sum_{i=1}^{n-1} \frac{x_i}{6^i}\right)\right\| \geq \frac{4}{5}\frac{\|T_n\|}{6^n} - \frac{2}{5}\frac{\|T_n\|}{6^n} = \frac{2}{5}\frac{\|T_n\|}{6^n}$$

Now, choose $x = \sum_{i=1}^{\infty} \frac{x_i}{6^i}$. By construction $\|x\| \leq 1$ and

$$\left\|T_n\left(\sum_{i=1}^{\infty} \frac{x_i}{6^i}\right)\right\| \geq \left\|T_n\left(\sum_{i=1}^{n} \frac{x_i}{6^i}\right)\right\| - \left\|T_n\left(\sum_{i=n+1}^{\infty} \frac{x_i}{6^i}\right)\right\| \geq \frac{2}{5}\frac{\|T_n\|}{6^n} - \frac{1}{5}\frac{\|T_n\|}{6^n} = \frac{1}{5}\frac{\|T_n\|}{6^n} \geq n$$

Therefore $\sup_{T \in \mathcal{T}} \|Tx\| = \infty$, which contradicts the hypothesis. ◻

Here we can see in a constructive way that, if the operators are unbounded, we can construct a point x such that $T_n x$ blows up when $n \to \infty$. The construction of x is made with a sum of terms which is exploding the fact that we can find operators as large (in boundedness sense) as we would like. We have to be careful because some terms may interfere with others.

This construction can be used when we have linear PDEs in unbounded domains so we can build initial datums that grow up or even blow up at some finite time.